New Results on Periodic Symbolic Sequences of Second Order Digital Filters with Two's Complement Arithmetic

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SUMMARY

In this article, the second order digital filter with two's complement arithmetic in [1] is considered. Necessary conditions for the symbolic sequences to be periodic after a number of iterations are given when the filter parameters are at $b = a + 1$ and $b = -a + 1$. Furthermore, for some particular values of a, even when one of the eigenvalues is outside the unit circle, the system may behave as a linear system after a number of iterations and the state vector may toggle between two states or converge to a fixed point at the steady state. The necessary and sufficient conditions for these phenomena are given in this article.

KEY WORDS: second order digital filter; two's complement arithmetic; symbolic sequences; eigenvalues

1. INTRODUCTION

A nonlinear behavior may occur on a second order digital filter when the filter is implemented using a two's complement arithmetic for the addition operation. To analyze such a behavior, a symbolic analysis was proposed and the admissibility of the symbolic sequences was studied for the special case when b=-1 and $|a|<2$ [1-4, 7]. However, even when the symbolic sequence is admissible, there are many possibilities. In order to study the various possibilities, the set of admissible sequences can be partitioned into three subsets: One set contains periodic symbolic sequences. The second set contains symbolic sequences that are periodic after a number of iterations. The third set contains symbolic sequences that are never periodic. Some results on the first two sets were obtained for b=-1 and $|a|<2$ [1-4, 7]. These results have been extended for other real values of a, while the value of b is still equal to –1 [5].

However, will those existing results remain valid if the filter parameter $b \neq -1$? Specifically, we are interested in the nonlinear behavior which may occur when $b = a+1$ and $b = -a+1$, and the answer to the following questions: Under what conditions will the symbolic sequence be periodic? If the symbolic sequence is periodic, under what conditions will the system behave as a linear system after a number of iterations and the state vector toggles among several states or converges to a fixed point? In this article, we focus on both the cases of $b = a+1$ and $b = -a+1$.

In section 2, we will present the notations used in the existing literatures [1-7], and this article will also employ the same set of notations. In section 3, some new results on the above problems are presented. Finally, a conclusion is summarized in section 4.

2. NOTATIONS

The notations used in [1-7] are adopted as follows:

The system is defined as:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \mathrm{F}\left(\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}\right) = \begin{bmatrix} x_2(k) \\ f(b \cdot x_1(k) + a \cdot x_2(k)) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B} \cdot s_k \tag{1}$$

where $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \in I^2 \equiv \{(x_1, x_2) : -1 \le x_1 < 1, -1 \le x_2 < 1\}$ (2)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} \tag{3}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \tag{4}$$

$s_k \in \{-m, \cdots, -1, 0, 1, \cdots m\}$ where m is the minimum integer satisfying

$$-2 \cdot m - 1 \le b \cdot x_1 + a \cdot x_2 < 2 \cdot m + 1 \tag{5}$$

and $f(x) = x - 2 \cdot n$ such that $2 \cdot n - 1 \le x < 2 \cdot n + 1$ and $n \in \mathrm{Z}^+ \cup \{0\}$ (6)

Given an initial condition $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \in I^2$, a symbolic sequence $s = (s_0, s_1, \cdots) \in \Sigma$ can be generated by the map $\mathrm{S} : I^2 \to \Sigma$, and a sequence s in Σ is admissible if $\exists \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \in I^2$ such that $\mathrm{S}\left(\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}\right) = s$.

The set Σ can be partitioned into three subsets: $\Sigma_\alpha = \{s = (s_0 s_1 s_2 \cdots) : s \text{ is periodic}\}$, $\Sigma_\beta = \{s = (s_0 s_1 s_2 \cdots) : s \text{ is periodic after a number of iterations}\}$ and $\Sigma_\gamma = \Sigma \setminus (\Sigma_\alpha \cup \Sigma_\beta)$.

3. NEW RESULTS FOR THE PERIODIC SYMBOLIC SEQUENCES

This section presents several conditions for the state vector to be periodic after a number of iterations. These conditions can be stated in the following lemmas, theorems and remarks, where the stability of these periodic orbits is stated in the observations:

Lemma 1

For $b=a+1$, *if* $\exists M \in Z^{+}$ such that $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} x_1(k+M) \\ x_2(k+M) \end{bmatrix}$ for $k \geq k_0$, *then*

$$(x_1(0)+x_2(0))\cdot\left((a+1)^M-1\right)+2\cdot\left(2\cdot(a+1)^M-1\right)\cdot\sum_{j=0}^{k_0-1}\frac{s_j}{(a+1)^{j+1}}+2\cdot(a+1)^M\cdot\sum_{j=k_0}^{k_0+M-1}\frac{s_j}{(a+1)^{j+1}}=0\cdot$$

Proof

For $b=a+1$, the nonlinear system in (1) can be represented by the state equation $\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a+1 & a \end{bmatrix}\cdot\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}+\begin{bmatrix} 0 \\ 2 \end{bmatrix}\cdot s_k$. Hence, the solution of the system is:

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}=\frac{1}{a+2}\cdot\begin{bmatrix} (a+1)^k+(a+1)\cdot(-1)^k & (a+1)^k-(-1)^k \\ (a+1)^{k+1}-(a+1)\cdot(-1)^k & (a+1)^{k+1}+(-1)^k \end{bmatrix}\cdot\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}+\frac{2}{a+2}\cdot\sum_{j=0}^{k-1}\begin{bmatrix} (a+1)^{k-1-j}-(-1)^{k-1-j} \\ (a+1)^{k-j}+(-1)^{k-1-j} \end{bmatrix}\cdot s_j \quad (7)$$

If $\exists M \in Z^{+}$ such that $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} x_1(k+M) \\ x_2(k+M) \end{bmatrix}$ for $k \geq k_0$, then:

$$\begin{aligned}&\frac{1}{a+2}\cdot\begin{bmatrix} (a+1)^{k_0}+(a+1)\cdot(-1)^{k_0} & (a+1)^{k_0}-(-1)^{k_0} \\ (a+1)^{k_0+1}-(a+1)\cdot(-1)^{k_0} & (a+1)^{k_0+1}+(-1)^{k_0} \end{bmatrix}\cdot\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}+\frac{2}{a+2}\cdot\sum_{j=0}^{k_0-1}\begin{bmatrix} (a+1)^{k_0-1-j}-(-1)^{k_0-1-j} \\ (a+1)^{k_0-j}+(-1)^{k_0-1-j} \end{bmatrix}\cdot s_j \\ &=\frac{1}{a+2}\cdot\begin{bmatrix} (a+1)^{k_0+M}+(a+1)\cdot(-1)^{k_0+M} & (a+1)^{k_0+M}-(-1)^{k_0+M} \\ (a+1)^{k_0+M+1}-(a+1)\cdot(-1)^{k_0+M} & (a+1)^{k_0+M+1}+(-1)^{k_0+M} \end{bmatrix}\cdot\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}+\frac{2}{a+2}\cdot\sum_{j=0}^{k_0+M-1}\begin{bmatrix} (a+1)^{k_0+M-1-j}-(-1)^{k_0+M-1-j} \\ (a+1)^{k_0+M-j}+(-1)^{k_0+M-1-j} \end{bmatrix}\cdot s_j\end{aligned} \quad (8)$$

Hence,

$$\begin{cases}\left((a+1)^M-1\right)\cdot\frac{(a+1)^{k_0}}{a+2}\cdot\left(x_1(0)+x_2(0)+2\cdot\sum_{j=0}^{k_0-1}\frac{s_j}{(a+1)^{j+1}}\right)+\left((-1)^M-1\right)\cdot\frac{(-1)^{k_0}}{a+2}\cdot\left((a+1)\cdot x_1(0)-x_2(0)-2\cdot\sum_{j=0}^{k_0-1}\frac{s_j}{(-1)^{j+1}}\right) \\ +\frac{2}{a+2}\cdot\sum_{j=0}^{k_0+M-1}s_j\cdot\left((a+1)^{k_0+M-1-j}-(-1)^{k_0+M-1-j}\right)=0\end{cases} \quad (9)$$

$$\begin{cases}\left((a+1)^M-1\right)\cdot\frac{(a+1)^{k_0+1}}{a+2}\cdot\left(x_1(0)+x_2(0)+2\cdot\sum_{j=0}^{k_0-1}\frac{s_j}{(a+1)^{j+1}}\right)+\left((-1)^M-1\right)\cdot\frac{(-1)^{k_0}}{a+2}\cdot\left(2\cdot\sum_{j=0}^{k_0-1}\frac{s_j}{(-1)^{j+1}}-\left((a+1)\cdot x_1(0)-x_2(0)\right)\right) \\ +\frac{2}{a+2}\cdot\sum_{j=0}^{k_0+M-1}s_j\cdot\left((a+1)^{k_0+M-j}+(-1)^{k_0+M-1-j}\right)=0\end{cases} \quad (10)$$

Let

$$t_1 = \left((a+1)^M - 1\right)\cdot\frac{(a+1)^{k_0}}{a+2}\cdot\left(x_1(0)+x_2(0)+2\cdot\sum_{j=0}^{k_0-1}\frac{s_j}{(a+1)^{j+1}}\right) \tag{11}$$

$$t_2 = \left((-1)^M - 1\right)\cdot\frac{(-1)^{k_0}}{a+2}\cdot\left((a+1)\cdot x_1(0)-x_2(0)-2\cdot\sum_{j=0}^{k_0-1}\frac{s_j}{(-1)^{j+1}}\right) \tag{12}$$

$$t_3 = \frac{2}{a+2}\cdot\sum_{j=0}^{k_0+M-1} s_j\cdot(a+1)^{k_0+M-1-j} \tag{13}$$

$$t_4 = \frac{2}{a+2}\cdot\sum_{j=0}^{k_0+M-1} s_j\cdot(-1)^{k_0+M-1-j} \tag{14}$$

Then we have:

$$\begin{cases} t_1+t_2+t_3-t_4=0 \\ (a+1)\cdot t_1 - t_2 + (a+1)\cdot t_3 + t_4 = 0 \end{cases} \tag{15}$$

which

$$\Rightarrow a=-2 \text{ or } t_1=-t_3 \tag{16}$$

For $a\neq -2$, we have $t_1=-t_3$, that is:

$$\left((a+1)^M - 1\right)\cdot\frac{(a+1)^{k_0}}{a+2}\cdot\left(x_1(0)+x_2(0)+2\cdot\sum_{j=0}^{k_0-1}\frac{s_j}{(a+1)^{j+1}}\right) = -\frac{2}{a+2}\cdot\sum_{j=0}^{k_0+M-1} s_j\cdot(a+1)^{k_0+M-1-j} \tag{17}$$

$$\Rightarrow (x_1(0)+x_2(0))\cdot\left((a+1)^M-1\right)+2\cdot\left(2\cdot(a+1)^M-1\right)\cdot\sum_{j=0}^{k_0-1}\frac{s_j}{(a+1)^{j+1}}+2\cdot(a+1)^M\cdot\sum_{j=k_0}^{k_0+M-1}\frac{s_j}{(a+1)^{j+1}}=0 \quad \blacksquare$$

Remark 1

If, after a number of iterations, the state vector is periodic with period *M*, then the symbolic sequence will also be periodic with the same period, that is, $s\in\Sigma_\beta$. Hence, this lemma gives a necessary condition for a symbolic sequence to be periodic after a number of iterations.

However, s_k is an integer in $\{-m,\cdots,-1,0,1,\cdots m\}$ and the periodicity of the symbolic sequence is *M*. So there are $(2\cdot m+1)^M$ possibilities. In the following theorem, the case of $s_k = 0$ for $k \ge k_0$ is discussed.

Theorem 1

For $b = a+1$ and $|a+1| > 1$, $s_k = 0$ for $k \ge k_0$ *if and only if* $\exists k_0 \in \mathrm{Z}^+ \cup \{0\}$ such that $x_1(k_0) = -x_2(k_0)$.

Proof

For the *if* part, since $b = a+1$ and $s_k = 0$ for $k \ge k_0$, we have:

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \frac{1}{a+2} \cdot \begin{bmatrix} (a+1)^{k-k_0} \cdot (x_1(k_0) + x_2(k_0)) + (-1)^{k-k_0} \cdot ((a+1) \cdot x_1(k_0) - x_2(k_0)) \\ (a+1)^{k-k_0+1} \cdot (x_1(k_0) + x_2(k_0)) - (-1)^{k-k_0} \cdot ((a+1) \cdot x_1(k_0) - x_2(k_0)) \end{bmatrix} \quad (18)$$

Since $|a+1| > 1$, $(a+1)^{k-k_0} \cdot (x_1(k_0) + x_2(k_0))$ diverges as $k \to +\infty$ if $x_1(k_0) + x_2(k_0) \ne 0$. However, as $s_k = 0$ for $k \ge k_0$, this implies that $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \in I^2$ and that $(a+1)^{k-k_0} \cdot (x_1(k_0) + x_2(k_0))$ is bounded for $k \ge k_0$. Hence, $x_1(k_0) + x_2(k_0) = 0$ and this proves the *if* part.

For the *only if* part, since $x_1(k_0) = -x_2(k_0)$, we have:

$\begin{bmatrix} x_1(k_0+1) \\ x_2(k_0+1) \end{bmatrix} = \begin{bmatrix} -x_1(k_0) \\ f((a+1) \cdot x_1(k_0) - a \cdot x_1(k_0)) \end{bmatrix} = \begin{bmatrix} -x_1(k_0) \\ f(x_1(k_0)) \end{bmatrix}$. Since $|x_1(k_0)| < 1$, we have $\begin{bmatrix} x_1(k_0+1) \\ x_2(k_0+1) \end{bmatrix} = \begin{bmatrix} -x_1(k_0) \\ x_1(k_0) \end{bmatrix}$ and $s_{k_0} = 0$.

Similarly, we have $\begin{bmatrix} x_1(k_0+2) \\ x_2(k_0+2) \end{bmatrix} = \begin{bmatrix} x_1(k_0) \\ -x_1(k_0) \end{bmatrix}$ and $s_{k_0+1} = 0$. Since $\begin{bmatrix} x_1(k_0+2) \\ x_2(k_0+2) \end{bmatrix} = \begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix}$, we have $s_k = 0$ for $k \ge k_0$ and this proves the *only if* part. ■

Remark 2

The eigenvalues of matrix $\mathbf{A}$ is -1 and $a+1$. Since $|a+1| > 1$, one of the eigenvalues is outside the unit circle. However, the system may behave as a linear system when $s_k = 0$ for $k \geq k_0$. Theorem 1 gives the necessary and sufficient condition for the nonlinear system to behave as a linear system after a number of iterations. It is interesting to note that the system may behave as a linear system after a number of iterations if and only if the state vector toggles between two points on a particular straight line of the phase portrait.

Example 1

Consider the system $\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B} \cdot s_k$, where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ a+1 & a \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $a = 3$ and $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0.9003 \\ -0.5377 \end{bmatrix}$. Figure 1 shows the phase portrait of the system. It can be seen from the figure that the state vector toggles between two states at the steady state on a particular straight line $x_1 = -x_2$ of the phase portrait.

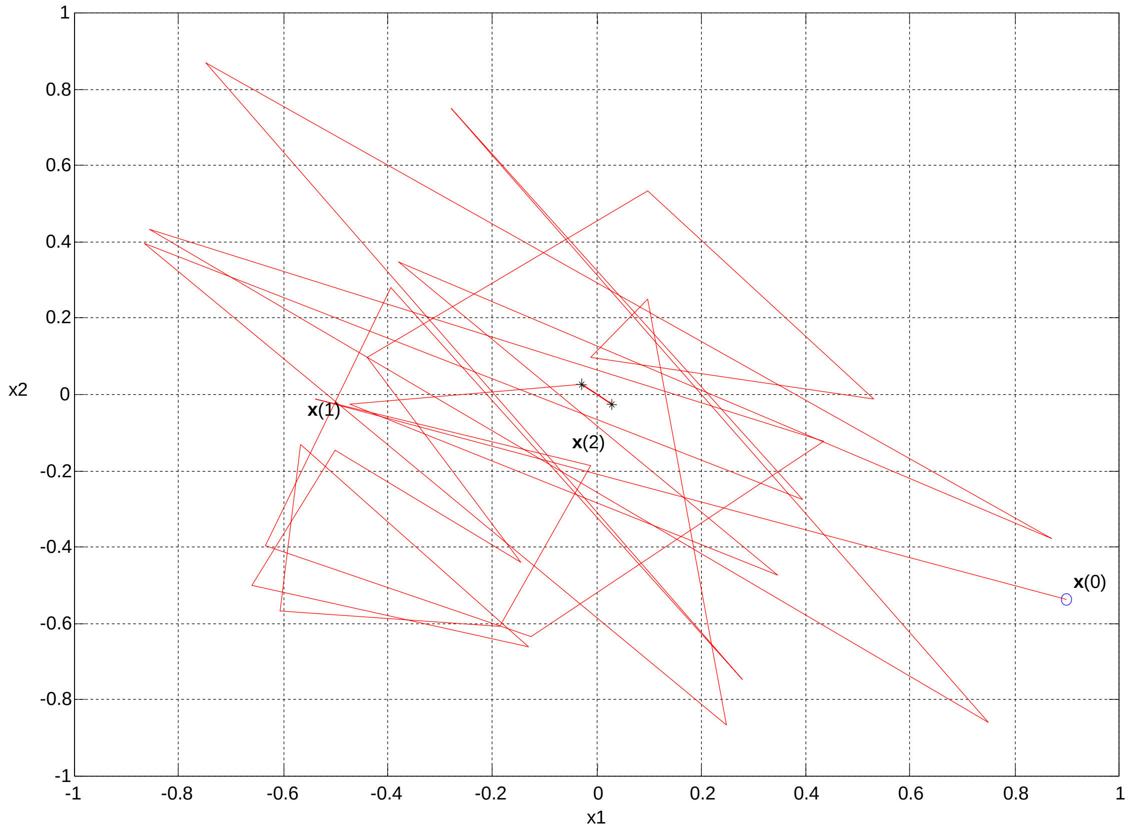

Figure 1. The phase portrait of the second order digital filter with two's complement arithmetic. The points **x**(0), **x**(1), **x**(2) are as annotated, and the points with '*' denote the 'steady states' of **x**.

We have discussed the case when $s_k = 0$ for $k \geq k_0$ in theorem 1. What happens when $s_k \neq 0$ for $k \geq k_0$? We will present an interesting result in the following theorem:

Theorem 2

For $b = a + 1$ and a being an odd integer, $\exists k_0 \in Z^+ \cup \{0\}$ such that $x_1(k_0) = x_2(k_0) = -1$ *if and only if* $s_k = a$ and $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix}$ for $k \geq k_0$.

Proof

For the *if* part, since $\begin{bmatrix} x_1(k_0+1) \\ x_2(k_0+1) \end{bmatrix} = \begin{bmatrix} x_2(k_0) \\ f((a+1)\cdot x_1(k_0) + a\cdot x_2(k_0)) \end{bmatrix}$, if $\exists k_0 \in Z^+ \cup \{0\}$ such that $x_1(k_0) = x_2(k_0) = -1$, then we have $\begin{bmatrix} x_1(k_0+1) \\ x_2(k_0+1) \end{bmatrix} = \begin{bmatrix} -1 \\ f(-(2\cdot a+1)) \end{bmatrix}$. Since a is an odd integer, we have $\begin{bmatrix} x_1(k_0+1) \\ x_2(k_0+1) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $s_{k_0} = a$. As $\begin{bmatrix} x_1(k_0+1) \\ x_2(k_0+1) \end{bmatrix} = \begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, we have $s_k = a$ and $x_1(k) = x_2(k) = -1$ for $k \geq k_0$.

For the *only if* part, since $\begin{bmatrix} x_1(k_0+1) \\ x_2(k_0+1) \end{bmatrix} = \begin{bmatrix} x_2(k_0) \\ f((a+1)\cdot x_1(k_0) + a\cdot x_2(k_0)) \end{bmatrix}$, if $x_1(k) = x_1(k_0)$ and $x_2(k) = x_2(k_0)$ for $k \geq k_0$, then we have $\begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} = \begin{bmatrix} x_2(k_0) \\ f((a+1)\cdot x_1(k_0) + a\cdot x_2(k_0)) \end{bmatrix}$, which implies $x_1(k_0) = x_2(k_0)$ and $x_2(k_0) = f((a+1)\cdot x_1(k_0) + a\cdot x_2(k_0))$. Since $s_k = a$ for $k \geq k_0$, we have $x_1(k_0) = (a+1)\cdot x_1(k_0) + a\cdot x_1(k_0) + 2\cdot a$, which implies $2\cdot a\cdot(x_1(k_0)+1) = 0$. As a is an odd integer, so $a \neq 0$. As a result, we have $x_1(k_0) = -1$ and we prove the *only if* part.■

Remark 3

Theorem 2 states the necessary and sufficient condition for the state vector to stay at a fixed point after a number of iterations when the parameter a is an odd integer. It is interesting to note that this fixed point is $(-1,-1)$.

Example 2

Consider the system $\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B} \cdot s_k$, where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ a+1 & a \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $a = 3$ and $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0.1875 \\ -0.25 \end{bmatrix}$. Figure 2 shows the phase portrait of the system. It can be seen from the figure that the state vector converges to a fixed point $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

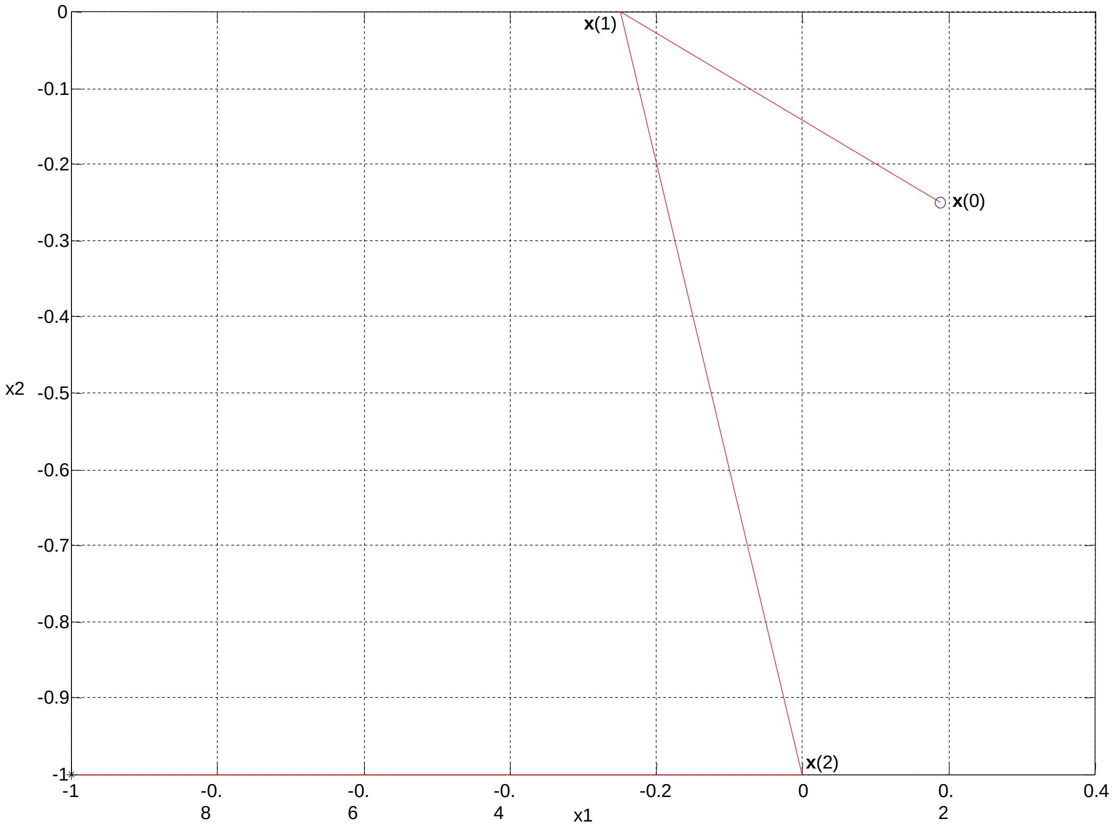

Figure 2. The phase portrait of the second order digital filter with two's complement arithmetic. The points **x**(0), **x**(1), **x**(2) are as annotated, and the point with '*' denotes the 'steady state' of **x**.

Remark 4

The above results are obtained when $b = a + 1$. What happens when $b = -a + 1$? We can show that results similar to lemma 1 and theorem 1 can be obtained as lemma 2 and theorem 3, while the results of theorem 2 can be modified to that of theorem 4.

Remark 5

Since one of the eigenvalues is unstable, one may predict that the periodic orbits are unstable. However, a counter-intuitive result is found that the periodic orbits are stable if a is an odd integer. The result is stated in observation 1 below:

Observation 1

When $b = a + 1$ and a is an odd integer, the state vector toggles between two states at the steady state on a particular straight line $x_1 = -x_2$ of the phase portrait or converges to a fixed point $\mathbf{x}^* = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ for all the initial conditions in I^2.

To demonstrate this phenomenon, a random initial condition $\mathbf{x}(0)$ is generated in I^2, it can be shown in figure 3a that the state converges to a period-2 signal.

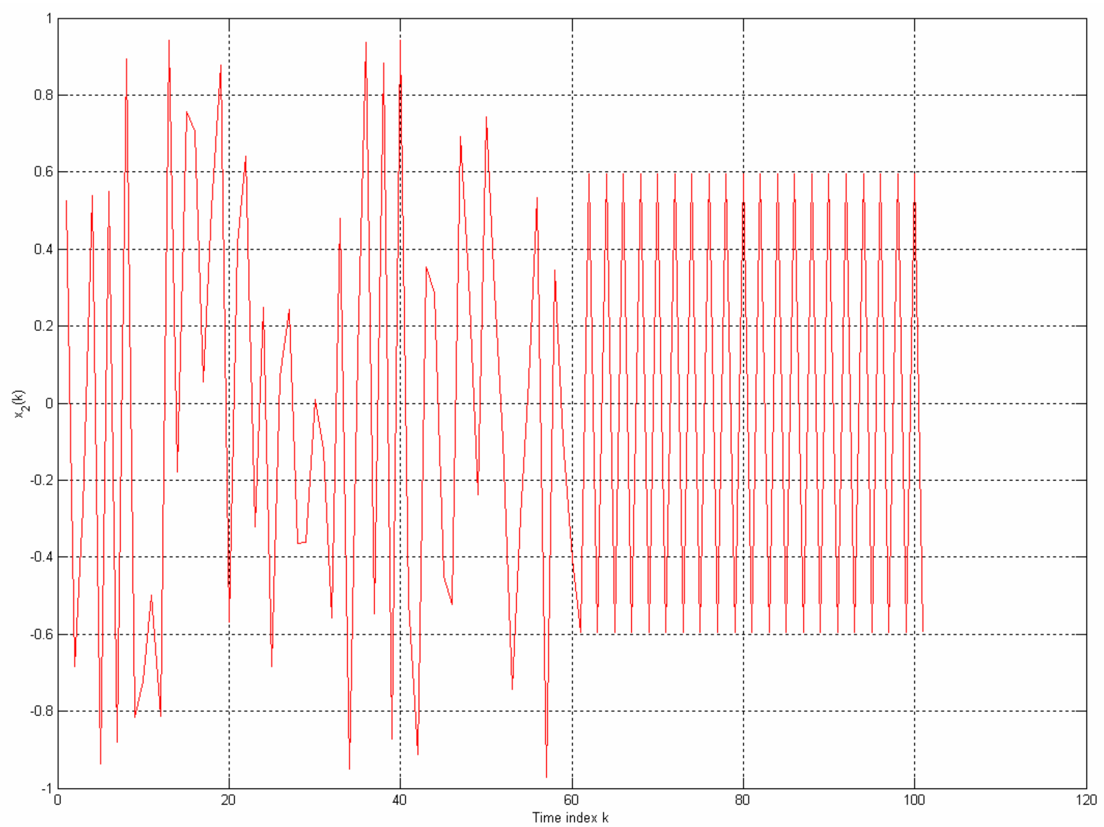

Figure 3a. The phase portrait of the second order digital filter with two's complement arithmetic. The initial condition $\mathbf{x}(0)=[0.7826\ 0.5242]^T$ is generated randomly. When a=5 and b=6, the state converges to a period-2 signal.

However, when a deviates from an odd integer a little bit, the state neither converges to a periodic signal nor a fixed point, as shown in figure 3b.

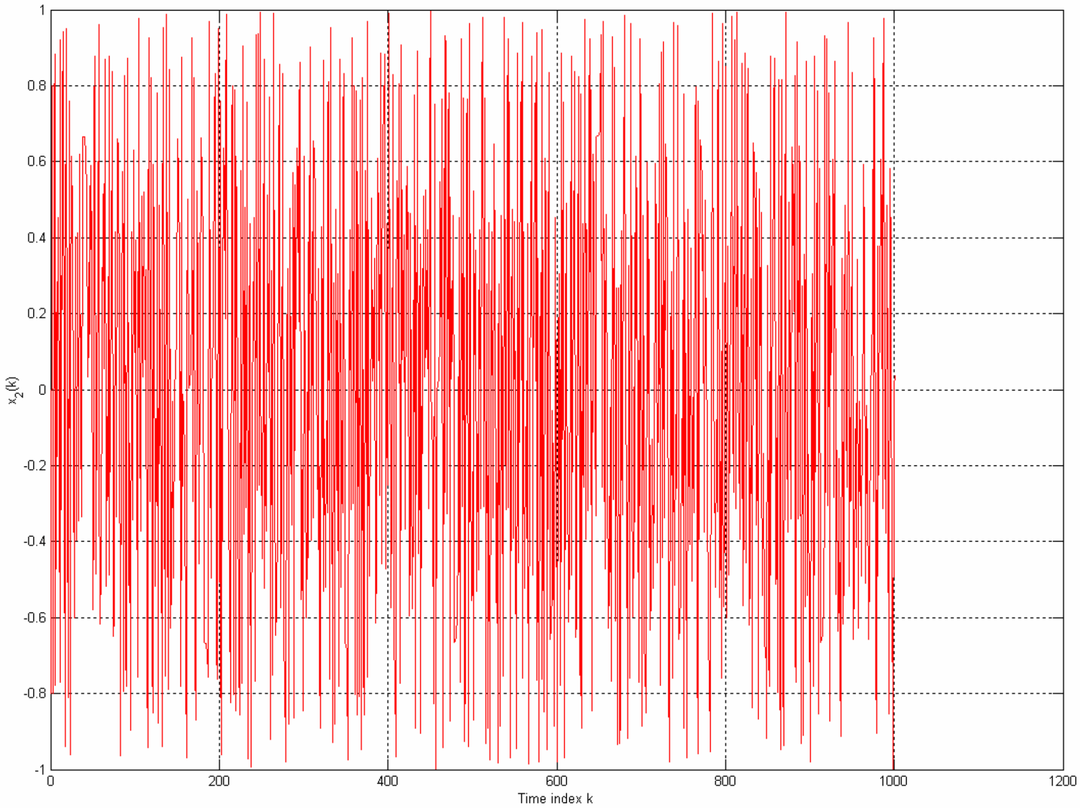

Figure 3b. The phase portrait of the second order digital filter with two's complement arithmetic. The initial condition is $\mathbf{x}(0)=[0.8\ -0.7999]^T$, a=3.001 and b=4.001, the state neither converges to a periodic signal nor a fixed point.

<u>Lemma 2</u>

For $b=-a+1$, *if* $\exists M \in Z^+$ such that $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} x_1(k+M) \\ x_2(k+M) \end{bmatrix}$ for $k \geq k_0$, *then*

$$\left(1-(a-1)^M\right)\cdot\left(x_1(0)-x_2(0)-2\cdot\sum_{j=0}^{k_0-1}\frac{s_j}{(a-1)^{j+1}}\right)+2\cdot\sum_{j=k_0}^{k_0+M-1}(a+1)^{M-j-1}\cdot s_j=0.$$

<u>Theorem 3</u>

For $b=-a+1$ and $|a-1|>1$, $s_k=0$ for $k \geq k_0$ *if and only if* $\exists k_0 \in Z^+ \cup \{0\}$ such that $x_1(k_0)=x_2(k_0)$.

<u>Example 3</u>

Consider the system $\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \mathbf{A}\cdot\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B}\cdot s_k$, where $\mathbf{A}=\begin{bmatrix} 0 & 1 \\ -a+1 & a \end{bmatrix}$,

$\mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $a = 3$ and $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -0.1875 \\ -0.1234 \end{bmatrix}$. Figure 4 shows the phase portrait of the system. It can be seen from the figure that the state vector converges to a fixed point on a particular straight line $x_1 = x_2$ of the phase portrait.

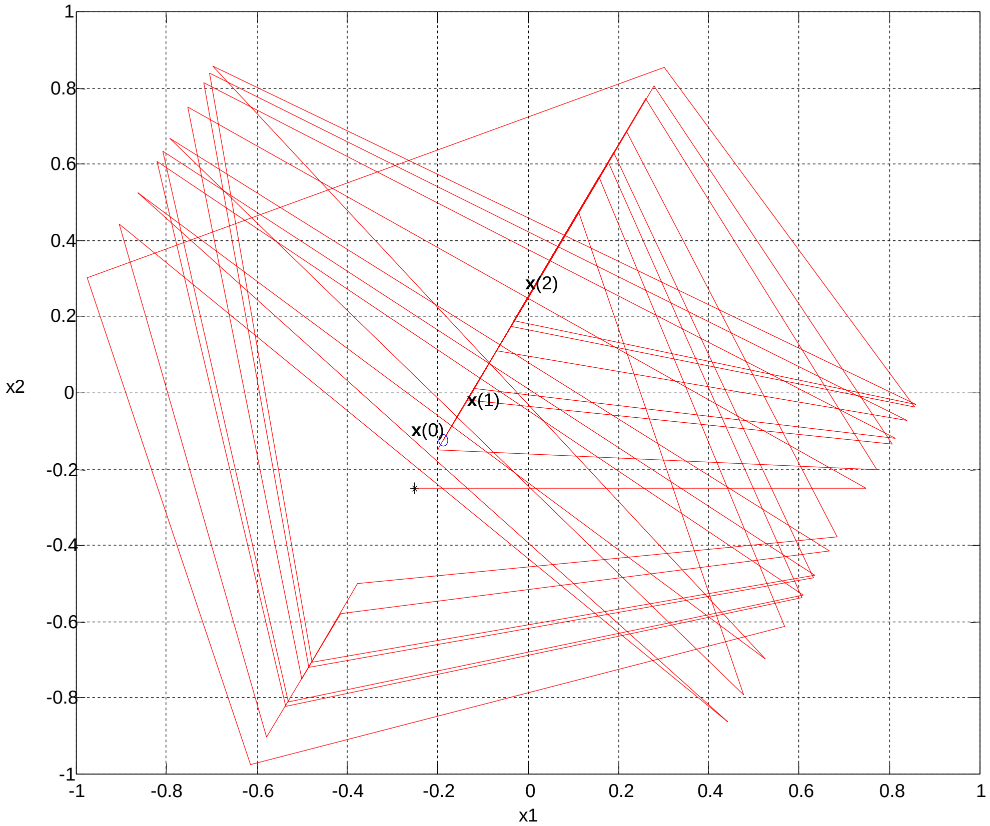

Figure 4. The phase portrait of the second order digital filter with two's complement arithmetic. The points **x**(0), **x**(1), **x**(2) are as annotated, and the point with '*' denotes the 'steady state' of **x**.

Theorem 4

For $b = -a + 1$ and a being an odd integer, there does not exist $k_0 \in Z^+$ such that $s_k = a$ for $k \geq k_0$.

Proof

Since $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \mathbf{A}^{k-k_0} \cdot \begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} + \sum_{j=k_0}^{k-1} \mathbf{A}^{k-1-j} \cdot \mathbf{B} \cdot s_j$ for $k > k_0$, if $s_k = a$ for $k \geq k_0$, we have:

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \frac{1}{a-2} \cdot \begin{bmatrix} (a-1)^{k-k_0} \cdot \left(x_2(k_0) - x_1(k_0) - \frac{2 \cdot a}{2-a} \right) + (a-1) \cdot x_1(k_0) - x_2(k_0) + \frac{2 \cdot a}{2-a} - 2 \cdot a \cdot (k - k_0) \\ (a-1)^{k-k_0+1} \cdot \left(x_2(k_0) - x_1(k_0) - \frac{2 \cdot a}{2-a} \right) + (a-1) \cdot x_1(k_0) - x_2(k_0) + \frac{2 \cdot a \cdot (a-1)}{2-a} - 2 \cdot a \cdot (k - k_0) \end{bmatrix} \quad (19)$$

As $k \to +\infty$, $k - k_0 \to +\infty$, so $\lim_{k \to +\infty} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \notin I^2$. Hence, there does not exist $k_0 \in Z^+$ such that $s_k = a$ for $k \geq k_0$, and this proves the theorem. ■

Example 4

Consider the system $\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B} \cdot s_k$, where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -a+1 & a \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $a = 3$ and $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0.7826 \\ 0.5242 \end{bmatrix}$. Figure 5 shows the phase portrait of the system.

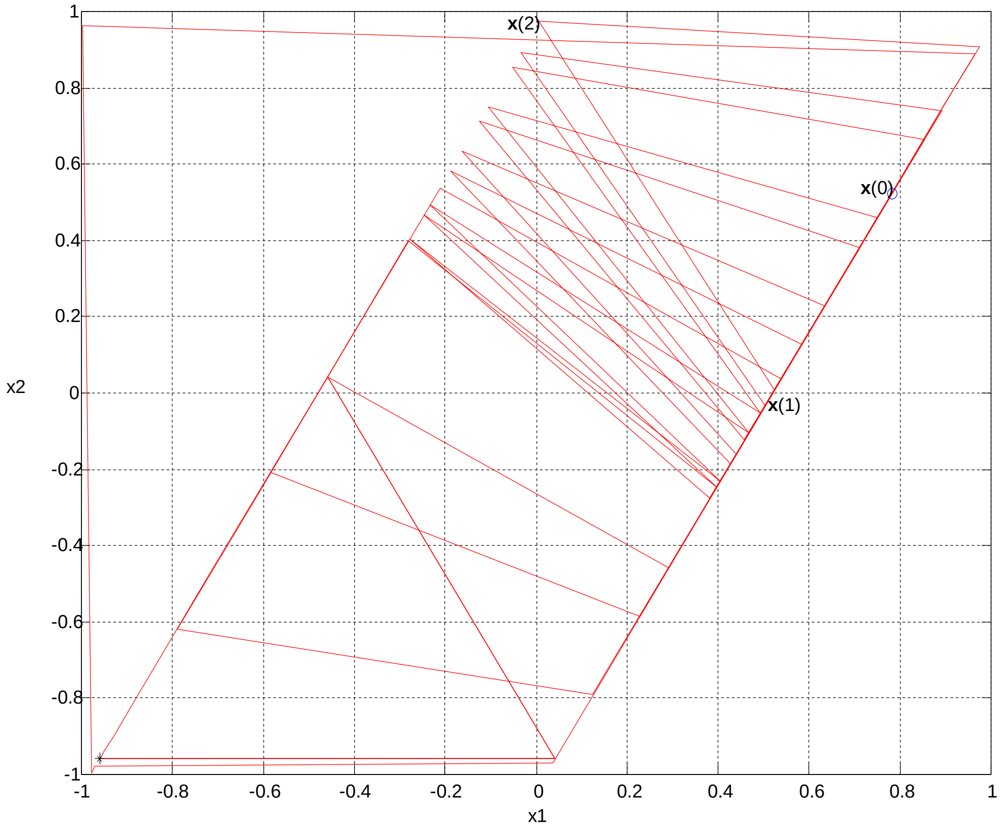

Figure 5. The phase portrait of the second order digital filter with two's complement arithmetic. The points **x**(0), **x**(1), **x**(2) are as annotated, and the point with '*' denotes the 'steady state' of **x**.

Observation 2

When $b = -a + 1$ and a is an odd integer, the state vector converges to a fixed point on a particular straight line $x_1 = x_2$ of the phase portrait for all the initial conditions in I^2.

To demonstrate this phenomenon, a random initial condition $\mathbf{x}(0)$ is generated in I^2, it can be shown in figure 6a that the state converges to a fixed point.

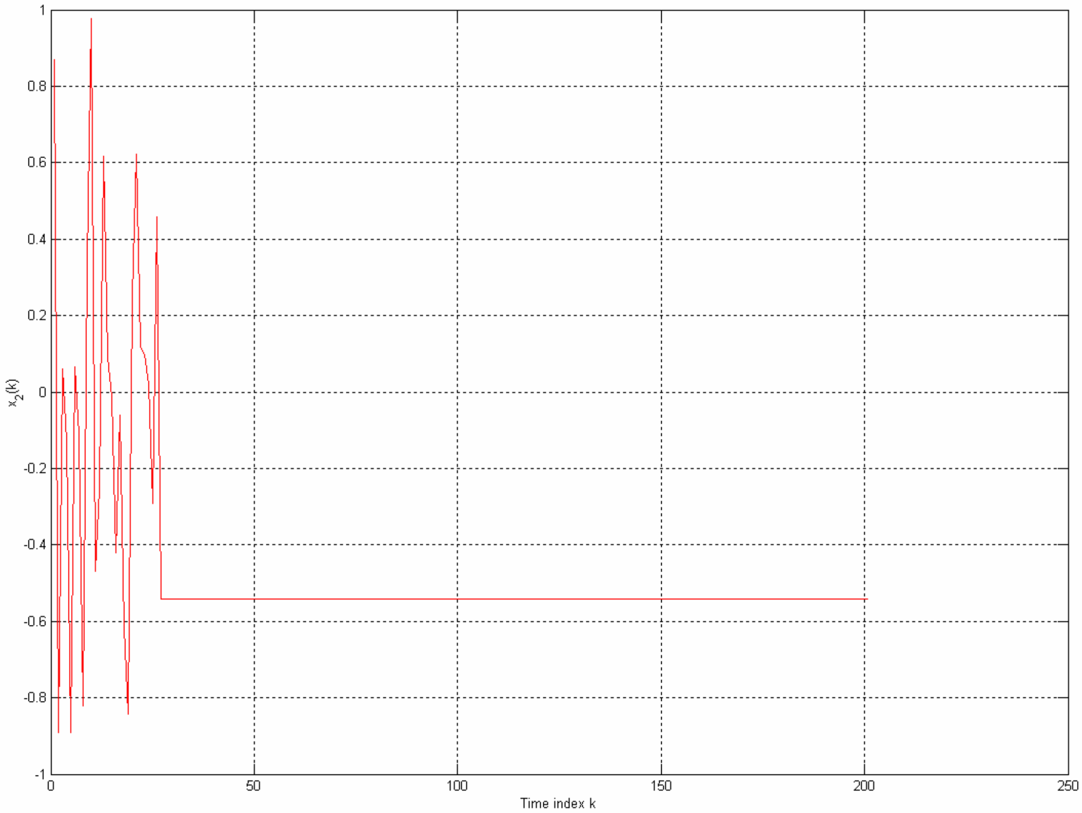

Figure 6a. The phase portrait of the second order digital filter with two's complement arithmetic. The initial condition is $\mathbf{x}(0)=[-0.1886\ 0.87909]^T$, a=5 and b=-4, the state converges to a fixed point.

However, when a deviates from an odd integer a little bit, the state does not converge to a fixed point, as shown in figure 6b.

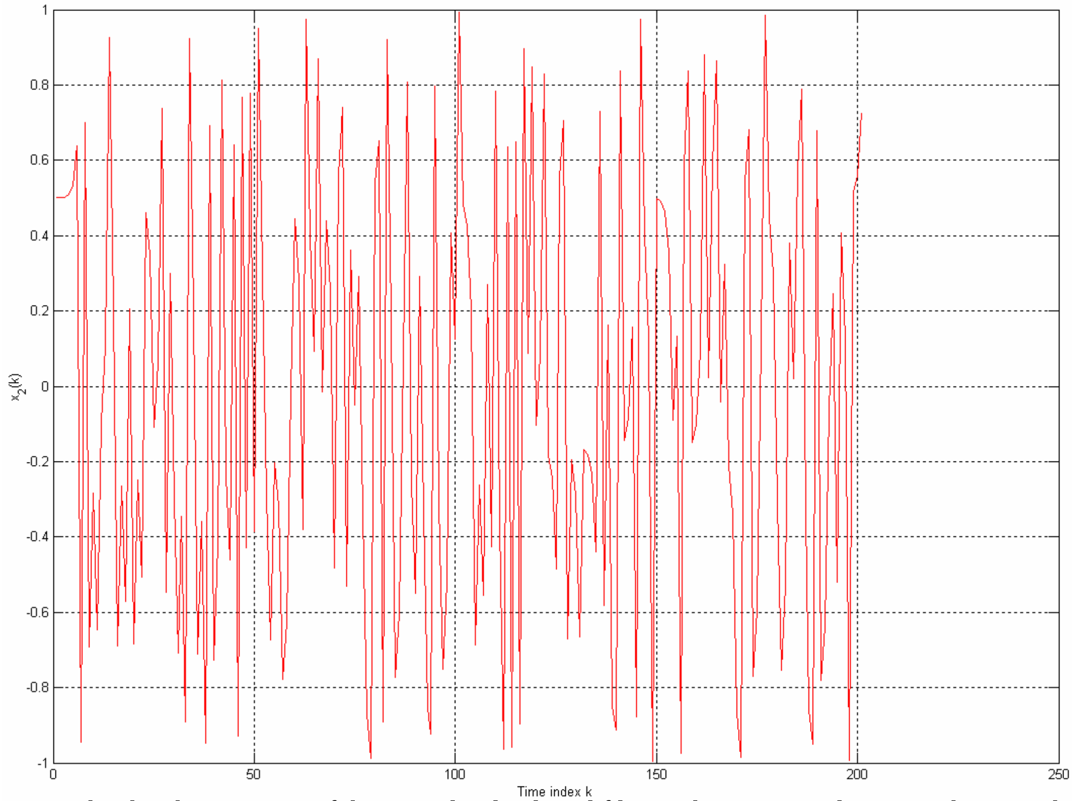

Figure 6b. The phase portrait of the second order digital filter with two's complement arithmetic. The initial condition is $\mathbf{x}(0)=[0.5\ 0.5001]^T$, a=5.01 and b=-4.01, the state does not converge to a fixed point.

4. CONCLUSIONS

In this article, some interesting behaviors of second-order digital filters with two's complement arithmetic are explored. The cases of $b = a + 1$ and $b = -a + 1$ are analyzed and some necessary conditions for the symbolic sequences to be periodic after a number of iterations are given. The necessary and sufficient conditions for the system to behave as a linear system after a number of iterations and the state vector to toggle among several states or converge to a fixed point are given.

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